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Schwarz type solvers for *hp*-FEM discretizations of mixed problems

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Schwarz type solvers for hp-FEM discretizations of mixed problems

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Abstract

The Stokes problem and linear elasticity problems can be viewed as a mixed variational formulation. These formulations are discretized by means of the hp-version of the finite element method. The system of linear algebraic equations is solved by the preconditioned Bramble-Pasciak conjugate gradient method. The development an efficient preconditioner requires three ingredients, a preconditioner related to the components of the velocity modes, a preconditioner for the Schur complementrelated to the components of the pressure modes and the discrezation by a stable finite element pair which satisfies the discrete inf-sup condition. The last condition is also important in order to obtain a stable discretization scheme. The preconditioner for the velocity modes is adapted from fast hp-FEM preconditioners for elliptic problems. Moreover, we will prove that the preconditioner for the Schur complement can be chosen as a diagonal matrix if the pressureis discretized by discontiuous finite elements.We will prove that the system of linear algebraic equationscan be solved in almost optimal complexity if the $Q_k - P_{k-1,disc}$ element is used. This yields to quasioptimal hp-FEM solvers for the Stokes problems and linear elasticity problems. The latter are robust with respect to the contraction ratio ν. The efficiency of the presented solver is shown in several numerical examples.

1 Introduction

The numerical solution of boundary value problems (BVP) of partial differential equations (PDE) is one of the major challenges in Computational Mathematics. Finite element methods (FEM) are among the most powerful tools in order to compute an approximate solution of BVP. For the h-version of the FEM, the polynomial degree k of the shape functions on the elements is kept constant and the mesh-size h is decreased. This is in contrast to the p-version of the FEM in which the polynomial degree k is increased and the mesh-size h is kept constant. Both ideas, mesh refinement and increasing the polynomial degree, can be combined. This is called the hp-version of the FEM. The advantage of the p-version in comparison to the h-version is that the solution converges much faster to the exact solution with respect to the dimension N of the approximation space, see e.g. [42], [43], [19] and the references therein as well as [30] for the related spectral element methods. For elliptic problems, preconditioned conjugate gradient (PCG) methods with additive Schwarz preconditioners (ASM) as domain decomposition (DD) are a powerful tool for the development of fast and efficient solvers for the h-version as well as for the p-version of the FEM, see [1, 5, 7, 8, 10, 17, 21, 27–29, 31, 33, 34]. The extension to linear elasticity problems is straightforward, [9]. Based on Korn's inequality, [22], an optimal preconditioner is obtained by using a preconditioner for the potential equation for each component of the displacement $\underline{u}$.

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In this paper, we will investigate the efficient solution of linear systems of algebraic equations of hp-FEM finite element discretizations of mixed formulations as, e.g., the Stokes problem. Other applications are robust solvers for linear elasticity. For nearly incompressible materials, the constants in Korn's inequality are very close to 0. In order to overcome this problem, a Lagrange multiplier $p = -\lambda\nabla \cdot u$ is introduced leading to a mixed problem. Mixed finite elements, see [16], are also used for saddle point problems such as the Stokes problem. An overview on flow problems is given in the monographs [38], [26]. The most important analytical tool for the development of a stable approximation scheme is the so called $\inf - \sup$ condition between the velocity $\underline{v}$, or the displacement $\underline{u}$, and the pressure p, which has to be verified for the corresponding pair of approximation spaces, see [14], [22]. In the h-version, stable element pairs are the Rannacher-Turek element, [39], the mini element, [45], the Q_2-Q_1-element and the elements with jumping pressure, [18]. In the p-version, the $Q_k - Q_{k-2}$ element is stable with respect to h, [44], see also [40], [41]. However, there is some dependence with respect to the polynomial degree k. The analysis for the related spectral element method has been done in [37], [36]. Another element is the $Q_k - P_{k-1,disc}$ element. Bernardi and Maday showed that the inf-sup-constant of a single element of this type is independent both of h and k [6]. With the macroelement technique of [41] one can then conclude that a whole mesh of elements of this type is inf-sup-stable independently of h and k. Using continuous pressure, Ainsworth and Coggins found an element which is inf-sup-stable uniformly with respect to both h and k for 2D [2] by using a truncated pressure space.
The discretization of the Stokes problem or the Lamé-equations in the mixed formulation leads to an indefinite system of linear algebraic equations, which can be solved by an UZAWA algorithm or GMRES. An alternative is the Bramble-Pasciak CG, [11, 46]. In this solution method, an inner product is defined in which the energetic inner product is positive definite. DD-methods for the Stokes problem have been considered in [13] for the h-version, in [4] for the p-version, and in [35], [25] for the related spectral element method.
The aim of this paper is the development of fast solvers for hp-FEM discretizations of mixed problem in $\mathcal{O}(N \log^{3/2} N)$ floating point operations, where N is the number of unknowns. The solvers use the preconditioned Bramble-Pasciak CG. The main ingredients of the solution method are an H^1 elliptic solver for the velocity part of the system, a inf-sup-stable finite element pair and a solver related to the mass matrix corresponding to the pressure modes. The solver for the velocity modes is an extension of the DD-based preconditioners in [8], [9] from elliptic problems. The stable $Q_k - P_{k-1,disc}$ element, [6], is the preferred finite element pair. Furthermore, we will show that the preconditioner for the pressure modes can be chosen as mass matrix.
The outline of the paper is follows. The setting of the problem is described in Section 2. The discretization with hp-finite elements is described in section 3. Section 4 deals with the numerical solution of the system of linear algebraic equations. Several numerical experiments are presented in Section 5.
Throughout this paper, the integer k denotes the polynomial degree. For two real symmetric and positive definite $n \times n$ matrices A, B, the relation $A \preceq B$ means that $A - cB$ is negative definite, where $c > 0$ is a constant independent of h, or k. The relation $A \sim B$ means $A \preceq B$ and $B \preceq A$, i.e. the matrices A and B are spectrally equivalent. The isomorphism between a function $u = \sum_i u_i\psi_i \in L^2$ and the vector of coefficients $\underline{u} = [u_i]_i$ with respect to the basis $[\Psi] = [\psi_1, \psi_2, \ldots]$ is denoted as $u = [\Psi]\underline{u}$.

2 Setting of the problem

In this paper, we consider the following problem. Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ be a bounded polygonal Lipschitz domain with $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\overline{\Gamma}_0 \cup \overline{\Gamma}_1 = \partial\Omega$, $meas(\Gamma_0) > 0$. Moreover, let

$$\mathbb{V} = H^1_{\Gamma_0}(\Omega, \mathbb{R}^d) = \{u \in H^1(\Omega, \mathbb{R}^d, u\mid_{\Gamma_0} = 0\} \quad \text{and} \quad \mathbb{Q} = \begin{cases} L^2_0(\Omega, \mathbb{R}^d) & \Gamma_1 = \emptyset \\ L^2(\Omega, \mathbb{R}^d) & \text{else} \end{cases},$$

where $L^2_0(\Omega, \mathbb{R}^d) = \{u \in L_2(\Omega, \mathbb{R}^d), \int_\Omega u \,\mathrm{d}x = 0\}$

Moreover, let

- $a(\cdot,\cdot)$ be a symmetric and bounded elliptic bilinear form on $\mathbb{V}$,
- for $v \in \mathbb{V}$ and $q \in \mathbb{Q}$, let $b(v,q) = -\int_\Omega q \operatorname{div} v \, \mathrm{d}x$, and
- $c(\cdot,\cdot)$ be symmetric and bounded positive semidefinite bilinear form on $\mathbb{Q}$.

Finally, let

$$F(v) := \int_\Omega f \cdot v \, \mathrm{d}x + \int_{\Gamma_1} g \cdot v \, \mathrm{d}s$$

with given $f \in L_2(\Omega, \mathbb{R}^d)$, $g \in L_2(\Gamma_1, \mathbb{R}^d)$. We are looking for solutions $(u,p) \in \mathbb{V} \times \mathbb{Q}$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= \langle F, v \rangle \quad \text{for all } v \in \mathbb{V}, &(1)\\ b(u,q) - c(p,q) &= 0 \quad \text{for all } q \in \mathbb{Q}. &(2) \end{aligned}$$

Two examples which are covered by this formulation are the Stokes problem and a mixed formulation of linear elasticity.
In the Stokes problem, we have

$$\begin{aligned} a(u,v) &= \nu \int_\Omega \operatorname{grad} u : \operatorname{grad} v \, \mathrm{d}x, &(3)\\ c(p,q) &= 0. \end{aligned}$$

Due to the existence of Dirichlet boundary conditions, the bilinear form $a(\cdot,\cdot)$ is $\mathbb{V}$ elliptic. Together with the inf-sup condition

$$\inf_{0 \neq q \in \mathbb{Q}} \sup_{0 \neq v \in \mathbb{V}} \frac{b(v,q)}{\|v\|_{\mathbb{V}} \|q\|_{\mathbb{Q}}} \geq \beta_1 > 0, \tag{4}$$

existence and uniqueness of the Stokes problem can be proved, see e.g. [26].
The mixed formulation for linear elasticity is obtained from the system of the Lamé equations

$$\begin{aligned} -2\mu \operatorname{div} \varepsilon(u) - \lambda \operatorname{grad} \operatorname{div} u &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Gamma_0,\\ \sigma(u) \cdot n &= g \quad \text{on } \Gamma_1, \end{aligned}$$

where λ and μ are the Lamé coefficients. Here, ε and σ are the strain tensor and stress tensor, respectively. Note that

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \quad \text{and} \quad \sigma = \lambda \operatorname{trace}(\varepsilon) I + 2\mu\varepsilon.$$

Finally, Γ_0 is the Dirichlet boundary and Γ_1 is the Neumann boundary.
The mixed formulation is obtained by introducing the hydrostatic pressure $p = -\lambda \operatorname{div} u$ as additional variable. Then, one obtains (1), (2) with the specifications:

$$\begin{aligned} a(u,v) &= 2\mu \int_\Omega \varepsilon(u) : \varepsilon(v) \, \mathrm{d}x, &(5)\\ c(p,q) &= \frac{1}{\lambda} \int_\Omega pq \, \mathrm{d}x. \end{aligned}$$

We note that from Korn's second inequality the following seminorm equivalence, see [32], [23], follows

$$c_K^2 |v|_{H^1}^2 \leq \int_\Omega \varepsilon(v) : \varepsilon(v) \, \mathrm{d}x \leq |v|_{H^1}^2, \tag{6}$$

where c_K is a constant depending on the geometry of Ω and on the Dirichlet boundary only. The inf-sup condition (4) guarantees existence and uniqueness of a weak solution $(u,p) \in \mathbb{V} \times \mathbb{Q}$, [16].

3 Galerkin discretization

The Stokes problem and the problem of linear elasticity are solved approximately by using Galerkin's method. Therefore by choosing appropriate finite-dimensional subspaces $\mathbb{V}_N \subset \mathbb{V}$, $\mathbb{Q}_N \subset \mathbb{Q}$ with $\dim(\mathbb{V}_N) + \dim(\mathbb{Q}_N) = N$ one can get an approximate solution of the above mentioned mixed variational problem (1), (2). The approximate solution solves the discrete variational problem

$$
\begin{aligned}
a(u_N, v_N) + b(v_N, p_N) &= F(v_N) \quad \text{for all} \quad v_N \in \mathbb{V}_N, && (7)\\
b(u_N, q_N) - c(p_N, q_N) &= 0 \quad \text{for all} \quad q_N \in \mathbb{Q}_N. && (8)
\end{aligned}
$$

In order to ensure existence and uniqueness of a weak solution $(p_N, q_N) \in \mathbb{V}_N \times \mathbb{Q}_N$ in (7), (8), the discrete inf-sup condition

$$
\inf_{0 \neq q_N \in \mathbb{Q}_N} \sup_{0 \neq v_N \in \mathbb{V}_N} \frac{b(v_N, q_N)}{\|v_N\|_{\mathbb{V}} \|q_N\|_{\mathbb{Q}}} \geq \tilde{\beta}_1 > 0 \tag{9}
$$

has to be satisfied. Then, the discrete mixed variational problem has a unique solution $(u_N, p_N) \in \mathbb{V}_N \times \mathbb{Q}_N$, [16]. Moreover, the error estimates depend strongly on $\tilde{\beta}_1$, e.g.

$$
\begin{aligned}
\|u - u_N\|_{\mathbb{V}} &\preceq \frac{1}{\tilde{\beta}_1} \inf_{v_N \in \mathbb{V}_N} \|u - v_N\|_{\mathbb{V}} + \inf_{q_N \in \mathbb{Q}_N} \|p - q_N\|_{\mathbb{Q}} \\
\|p - p_N\|_{\mathbb{Q}} &\preceq \frac{1}{\tilde{\beta}_1^2} \inf_{v_N \in \mathbb{V}_N} \|u - v_N\|_{\mathbb{V}} + \frac{1}{\tilde{\beta}_1} \inf_{q_N \in \mathbb{Q}_N} \|p - q_N\|_{\mathbb{Q}},
\end{aligned}
$$

see again [16] concerning the details. Therefore, an independence of $\tilde{\beta}_1 > 0$ of the discretization parameters is important in order to get a stable approximation scheme. Note that in general the existence of a discrete inf-sup-constant (9) does not follow from the continuous one (4). Therefore this condition has to be shown explicitly for each choice of $\mathbb{V}_N$ and $\mathbb{Q}_N$.

Now, we are able to describe the setup for the discrete problem. Ω is a polygonal Lipschitz domain, which is triangulated into a triangulation $\mathcal{T}_s$ consisting of isotropic quadrilateral (2D) or hexahedral (3D) elements $\mathcal{R}_s$. With $\mathcal{R}$ we denote the reference element $(-1,1)^d$ with $d = 2, 3$. Φ_s is a mapping from the reference element $\mathcal{R}$ to the element $\mathcal{R}_s$. $\mathcal{Q}_k$ denotes the space of polynomials on $(-1,1)^d$ with degree less then k in each variable whereas $\mathcal{P}_k$ denotes the space of polynomials on $(-1,1)^d$ with total degree less then k. For the velocity u we can now build the following hp-FEM space

$$
\mathbb{V}_k \quad = \quad \{u \in \mathbb{V}, \quad u|_{\mathcal{R}_s} = \tilde{u} \circ \Phi_s^{-1}, \tilde{u} \in (\mathcal{Q}_k)^d\}. \tag{10}
$$

Several choices for $\mathbb{Q}_N$ are considered as approximation space for the pressure. In this paper, the $Q_k - P_{k-1,disc}$, $Q_k - Q_{k-2,disc}$, the Taylor-Hood element and the Q_k-$Q'_{k-1,cont}$-element (only for $d = 2$) are investigated. The corresponding pressure spaces are defined by the relations

$$
\begin{aligned}
\mathbb{P}_{k-1} &= \{p \in \mathbb{Q}, \quad p|_{\mathcal{R}_s} = \tilde{p} \circ \Phi_s^{-1}, \tilde{p} \in \mathcal{P}_{k-1}\}, \\
\mathbb{Q}_{k-2} &= \{p \in \mathbb{Q}, \quad p|_{\mathcal{R}_s} = \tilde{p} \circ \Phi_s^{-1}, \tilde{p} \in \mathcal{Q}_{k-2}\}, \\
\mathbb{Q}_{k-1,cont} &= \{p \in \mathbb{Q}, p \in C^0(\Omega), \quad p|_{\mathcal{R}_s} = \tilde{p} \circ \Phi_s^{-1}, \tilde{p} \in \mathcal{Q}_{k-1}\}, \\
\mathbb{Q}'_{k-1,cont} &= \{p \in \mathbb{Q}, p \in C^0(\Omega), \quad p|_{\mathcal{R}_s} = \tilde{p} \circ \Phi_s^{-1}, \tilde{p} \in \mathcal{Q}'_{k-1}\}
\end{aligned} \tag{11}
$$

with

$$
\mathcal{Q}'_k = P_k \cup \operatorname{span}\{x_i y^j : i \in \{0,1\}, 0 \leq j \leq k \quad \text{or} \quad 0 \leq i \leq k, j \in \{0,1\}\},
$$

respectively, see [44], [3], [2]. The following lemma summarizes the behavior of the inf-sup constants.

Lemma 3.1. *Let $\mathbb{V}_k$, $\mathbb{P}_{k-1}$, $\mathbb{Q}_{k-2}$, $\mathbb{Q}_{k-1,cont}$ and $\mathbb{Q}'_{k-1,cont}$ be defined by* (10) *and* (11)*, respectively. Then,*

$$\begin{array}{rcl}
0 < \tilde{\beta}_1 & \preceq & \inf\limits_{0 \neq q_N \in \mathbb{P}_{k-1}} \sup\limits_{0 \neq v_N \in \mathbb{V}_k} \dfrac{b(v_N, q_N)}{\|v_N\|_{\mathbb{V}} \|q_N\|_{\mathbb{Q}}} \\
0 < \tilde{\beta}_2(k) & \preceq & \inf\limits_{0 \neq q_N \in \mathbb{Q}_{k-2}} \sup\limits_{0 \neq v_N \in \mathbb{V}_k} \dfrac{b(v_N, q_N)}{\|v_N\|_{\mathbb{V}} \|q_N\|_{\mathbb{Q}}} \\
0 < \tilde{\beta}_3(k) & \preceq & \inf\limits_{0 \neq q_N \in \mathbb{Q}_{k-1,cont}} \sup\limits_{0 \neq v_N \in \mathbb{V}_k} \dfrac{b(v_N, q_N)}{\|v_N\|_{\mathbb{V}} \|q_N\|_{\mathbb{Q}}} \\
0 < \tilde{\beta}_4 & \preceq & \inf\limits_{0 \neq q_N \in \mathbb{Q}'_{k-1,cont}} \sup\limits_{0 \neq v_N \in \mathbb{V}_k} \dfrac{b(v_N, q_N)}{\|v_N\|_{\mathbb{V}} \|q_N\|_{\mathbb{Q}}}
\end{array}$$

Proof. The result for $\mathbb{P}_{k-1}$ has been shown in [44]. Furthermore, the inf-sup-constant $\tilde{\beta}_1$ of a single element of this type is independent both of h and k, see [6]. With the macroelement technique in [41] one can then conclude that a whole mesh of elements of this type is inf-sup-stable independently of h and k. The result for $\mathbb{Q}_{k-2}$ has been proved in [44]. However, $\tilde{\beta}_2$ is not independent of the polynomial degree k. More precisely,

$$\tilde{\beta}_2 \succeq k^{-\frac{d-1}{2}}$$

with C independent of k and h, see [44]. The Taylor-Hood element is inf-sup-stable with respect to h [15]. However, the inf-sup-constant $\tilde{\beta}_3$ degrades with k [3]. The result for $\mathbb{Q}'_{k-1,cont}$ has been proved in [2]. □

In a next step, the spaces are equipped with basis functions. The Legendre polynomials on $(-1, 1)$ are defined as

$$L_i(x) = \frac{1}{2^i i!} \frac{\mathrm{d}^i}{\mathrm{d}x^i} \left(x^2 - 1\right)^i. \tag{12}$$

They form a orthogonal system, e.g.

$$\int_{-1}^{1} L_i(x) L_j(x)\ \mathrm{d}x = \delta_{ij} \frac{2}{2i-1}. \tag{13}$$

Shape functions for the 2D and 3D reference element of the discontinuous pressure spaces $\mathbb{P}_{k-1}$ and $\mathbb{Q}_{k-2}$, can now be constructed by taking products, i.e.

$$\begin{array}{rcl}
L_{ij}(x,y) & = & L_i(x)L_j(y) \quad 0 \leq i, j, \\
L_{ijr}(x,y,z) & = & L_i(x)L_j(y)L_r(z) \quad 0 \leq i, j, r
\end{array} \tag{14}$$

where the indices i, j and r are running to some upper limit depending on the maximal polynomial degree k and the choice of the elements for the pressure. Since the pressure lives in L^2 and therefore continuity across element boundaries is not necessary these polynomials can be used. The basis $[\Phi_{disc,p}] = [\phi_1^p, \ldots \phi_m^p,]$ is introduced as the set of all piecewise discontinuous Legendre polynomial functions under the mapping F_s. The support of each function consists of one element only.
However, for the velocity u and the pressure in the spaces $\mathbb{Q}_{k-1,cont}$ and $\mathbb{Q}'_{k-1,cont}$ continuity across element boundaries is required. Therefore, the integrated Legendre polynomials are introduced which are easier to handle:

$$\widehat{L}_i(x) = \gamma_i \int_{-1}^{x} L_{i-1}(s)\ \mathrm{d}s \quad i \geq 2 \qquad \text{and} \qquad \widehat{L}_{0/1}(x) = \frac{1 \pm x}{2} \tag{15}$$

with some scaling factor γ_i. Shape functions for 2D and 3D can now be constructed again by taking products

$$\begin{aligned}\widehat{L}_{ij}(x,y) &= \widehat{L}_i(x)\widehat{L}_j(y) \quad 0 \le i,j \le k,\\ \widehat{L}_{ijm}(x,y,z) &= \widehat{L}_i(x)\widehat{L}_j(y)\widehat{L}_m(z) \quad 0 \le i,j,m \le k,\end{aligned}$$

on the reference element, respectively. Since $\widehat{L}_i(\pm 1) = 0$ for $i \ge 2$, the global basis functions for the continuous pressure spaces $\mathbb{Q}_{k-1,cont}$ and $\mathbb{Q}'_{k-1,cont}$ are constructed in the usual way. The global functions ψ_i can be divided into four groups,

- the vertex functions,
- the edge bubble functions,
- face bubble functions,
- the interior bubble functions,

locally on each element $\mathcal{R}_s$, and globally on Ω. We denote them again with $[\Phi_{cont,p}] = [\phi_1^p, \dots, \phi_m^p]$. In the case of pure Dirichlet boundary conditions the space has to be modified since $\mathbb{Q}_N \subset L_0^2(\Omega)$. Since $\int_{-1}^1 \hat{L}_i(x)\ \mathrm{d}x = 0$ and $\int_{-1}^1 L_i(x)\ \mathrm{d}x = 0$ for $i \ge 2$, this modification does not affect the high order basis functions. We refer the interested reader to [24].
The basis functions for the velocity are vector valued functions with values in $\mathbb{R}^d$. They can be obtained from $[\Phi_{cont,p}]$ in the following way. Let $\mathcal{M} = \{i, i = 1, \dots, m, \phi_i^p \mid_{\Gamma_0} = 0\}$ be the set of all indices corresponding to basis functions vanishing at the Dirichlet boundary condition and

$$[\Phi_{u,1}] = [\phi_i^p]_{i \in \mathcal{M}} := [\psi_i^p]_i \,. \tag{16}$$

Then,

$$[\Phi_u] = [\phi_1^u, \dots, \phi_n^u] := \left[\begin{bmatrix} \psi_i^p \\ 0 \\ 0 \end{bmatrix}_i, \begin{bmatrix} 0 \\ \psi_i^p \\ 0 \end{bmatrix}_i, \begin{bmatrix} 0 \\ 0 \\ \psi_i^p \end{bmatrix}_i \right] \tag{17}$$

are the basis functions for the velocity for $d = 3$. The definition for $d = 2$ is similar.
Let

$$\begin{aligned} A = (a_{ij})_{i,j=1}^n &= \left[a\left(\phi_i^u, \phi_j^u\right)\right]_{i,j=1}^n, \\ B = (b_{ij})_{i,j=1}^{n,m} &= \left[b\left(\phi_i^u, \phi_j^p\right)\right]_{i,j=1}^{n,m}, \\ C = (c_{ij})_{i,j=1}^m &= \left[c\left(\phi_i^p, \phi_j^p\right)\right]_{i,j=1}^m. \end{aligned} \tag{18}$$

Then, the formulation (7), (8) is equivalent to the solution of a linear system of algebraic equations

$$K \begin{bmatrix} \underline{u_N} \\ \underline{p_N} \end{bmatrix} = \begin{bmatrix} \underline{f_N} \\ 0 \end{bmatrix} := \begin{bmatrix} A & B^\top \\ B & -C \end{bmatrix} \begin{bmatrix} \underline{u_N} \\ \underline{p_N} \end{bmatrix} = \begin{bmatrix} \underline{f_N} \\ 0 \end{bmatrix}, \tag{19}$$

where $\underline{f_h} = [f_j]_{j=1}^N$ with $f_j = F(\phi_j^u)_{j=1}^n$. Note that for the Stokes problem we have $C = 0$. Using the FE isomorphism, the approximate solutions u_N and p_N are obtained as $u_N = [\Phi_u]\underline{u_N}$ and $p_N = [\Phi_{disc,p}]\underline{p_N}$ or $p_N = [\Phi_{cont,p}]\underline{p_N}$, respectively.

4 Solution of the linear system

This subsection is devoted to the solution of (19). The matrix A comes from the discretization of the velocity or displacement part and is symmetric positive definite. Since the matrix B has full rank, the whole matrix K is non-singular. Moreover, the Schur complement $S = C + BA^{-1}B^\top$ is also symmetric positive definite. Since the system matrix $K = \begin{bmatrix} A & B^\top \\ B & -C \end{bmatrix}$ is indefinite, one can not use the preconditioned conjugate gradient (PCG) method for the solution of the system. Due to the structure of K, we are able to use the Bramble-Pasciak-CG, see [12].
In detail, let $\hat{A}$ and $\hat{S}$ be preconditioners for A and the Schur complement S with

$$\underline{\gamma_0}\hat{A} \le A \le \overline{\gamma_0}\hat{A} \quad \text{and} \quad \underline{\gamma_1}\hat{S} \le S \le \overline{\gamma_1}\hat{S}, \tag{20}$$

respectively, where $\underline{\gamma_0} > 1$. The matrix K in (19) is formally preconditioned by the matrix

$$L^{-1} = \begin{bmatrix} \hat{A} & 0 \\ B & -\hat{S} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{A}^{-1} & 0 \\ \hat{S}^{-1}B\hat{A}^{-1} & -\hat{S}^{-1} \end{bmatrix}$$

in the non standard inner product $(\cdot\,,\cdot)_H = (H\cdot\,,\cdot)$ with $H = \begin{bmatrix} A - \hat{A} & 0 \\ 0 & \hat{S} \end{bmatrix}$. In other words, the multiplication of (19) with HL^{-1} from left yields to

$$T \begin{bmatrix} \underline{u_N} \\ \underline{p_N} \end{bmatrix} = \begin{bmatrix} (A - \hat{A})\hat{A}^{-1}f \\ B\hat{A}^{-1}f \end{bmatrix}$$

with the matrix

$$T := HL^{-1}K = \begin{bmatrix} (A - \hat{A})\hat{A}^{-1}A & (A - \hat{A})\hat{A}^{-1}B^\top \\ B\hat{A}^{-1}(A - \hat{A}) & B\hat{A}^{-1}B^\top \end{bmatrix}. \tag{21}$$

This matrix is symmetric positive definite with respect to the standard inner product. Therefore, the PCG-method can be used with the preconditioner

$$\hat{T} = \begin{bmatrix} A - \hat{A} & 0 \\ 0 & \hat{S} \end{bmatrix}. \tag{22}$$

We have the following theorem, [12], see also [46] for improved estimates.

Theorem 4.1. *Let T be defined by* (21). *Let us assume that A is symmetric positive definite and B has maximal rank. Moreover, let $\hat{A}$ and $\hat{S}$ be preconditioners for A and the Schur complement S satisfying* (20). *Let $\hat{T}$ be defined by* (22). *Then, the condition number estimates*

$$\min\{1, \underline{\gamma_1}\}\underline{\gamma}\hat{T} \le T \le \max\{1, \overline{\gamma_1}\}\overline{\gamma}\hat{T}$$

hold where $\underline{\gamma} = \frac{1-\sqrt{\alpha}}{1-\alpha}$, and $\overline{\gamma} = \frac{1+\sqrt{\alpha}}{1-\alpha}$ with $\alpha = 1 - \frac{1}{\underline{\gamma_0}}$.

Summarizing, $\hat{T}$ is a good preconditioner for T if and only if $\hat{A}$ and $\hat{S}$ are good preconditioners for A and S respectively, cf. (20). In the next two subsections, these preconditioners are defined. The final condition number estimates are stated in subsection 4.3.

4.1 The preconditioner for the velocity part

This preconditioner corresponds to the bilinear form $a(\cdot,\cdot)$ which is symmetric and bounded elliptic bilinear form on $\mathbb{V} \subset H^1(\Omega, \mathbb{R}^d)$.
We use an overlapping DD-preconditioner with inexact subproblem solvers which has been developed for scalar elliptic problems and a scalar hp-FEM basis, see [9] for more details. Let

$$\mathfrak{A} = \int_\Omega \nabla[\Phi_{u,1}] \cdot \nabla[\Phi_{u,1}] \, \mathrm{d}x \tag{23}$$

be the corresponding stiffness matrix for the Laplacian with respect to the scalar basis $[\Phi_{u,1}] = [\psi_i^p]_i$ (16). The domain decomposition splitting is taken from [34].
We give only a very brief definition of the preconditioner. For a given node $\mathtt{v}$, let $\Omega_\mathtt{v} = \{\cup_s \overline{\mathcal{R}_s}, \mathtt{v} \subset \overline{\mathcal{R}_s}\}$ be the closed patch associated to a node $\mathtt{v}$ of the finite element mesh. Let $J(\mathtt{v}) = \left[j_1^\mathtt{v}, \ldots, j_{n_\mathtt{v}}^\mathtt{v}\right]$ be the index set of all basis functions with $\mathrm{supp}(\psi_j^u) \subset \Omega_\mathtt{v}$ and $J(0)$ the index set of all m_v global vertex functions (V) which are ordered first. Let $P_\mathtt{v} \in \mathbb{R}^{n_\mathtt{v} \times N}$ be the Boolean matrices with the entries

$$[P_\mathtt{v}]_{ij} = \begin{cases} 1 & \text{if} \quad j = j_i^\mathtt{v}, 1 \le i \le n_\mathtt{v} \\ 0 & \text{else} \end{cases} \quad \text{and} \quad [P_0]_{ij} = \begin{cases} 1 & \text{if} \quad i = j \le m_v \\ 0 & \text{else} \end{cases} .$$

Finally, let

$$\mathcal{C}_\mathtt{v} = \left[a(\psi^u_{j,i^\mathtt{v}}, \psi^u_{j,k^\mathtt{v}})\right]_{i,k=1}^{n_\mathtt{v}} . \tag{24}$$

In the same way, C_0 corresponding to the set $J(0)$ are introduced. Finally, the ASM preconditioner with inexact subproblem solvers for $\mathfrak{A}$ (23) is defined by choosing the BPX-preconditioner C_{BPX} and the matrix $\hat{C}_{3,p}$ as preconditioner for C_0 and $\mathcal{C}_\mathtt{v}$, respectively. The matrix $\hat{C}_{3,p}$ is the wavelet based preconditioner developed in [9, (22)] for the patches, see also [8] for more details. Summarizing, one obtains the preconditioner

$$\hat{A}_{in,1}^{-1} = P_0^\top C_{BPX}^{-1} P_0 + \sum_\mathtt{v} P_\mathtt{v}^\top \hat{C}_{3,p}^{-1} P_\mathtt{v}. \tag{25}$$

Since the velocity has values in $\mathbb{R}^d$, the preconditioner (25) has to be adapted

$$\hat{A}_{in,d} = \mathrm{blockdiag}\left[\hat{A}_{in,1}\right]_{i=1}^{d} . \tag{26}$$

Then, the next theorem 4.3 can be shown under the following assumption.

Assumption 4.2. *Each patch $\Omega_\mathtt{v}$ corresponding to an interior node for $d = 3$ is the union of eight hexahedrons, two in each space direction. Patches to Neumann nodes on faces or interior nodes for $d = 2$ of Ω are assumed to be the union of four hexahedrons.*

Theorem 4.3. *Let $a(\cdot,\cdot)$ be a symmetric, bounded and elliptic bilinear form on $\mathbb{V}$. Let A and $\hat{A}_{in,d}$ be defined by* (19) *and* (26)*, respectively. Moreover, let us assume that Assumption 4.2 is satisfied. Then, the condition number estimate $\kappa(\hat{A}_{in,d}^{-1} A) \preceq (\log k \log^\chi \log k)^3$ holds for any $\chi > 1$ where the constant is independent on h and k. Moreover, the action $\hat{A}_{in,d}^{-1} \underline{r}$ requires $\mathcal{O}(N)$ operations.*

Proof. The result $\kappa(\hat{A}_{in,1}^{-1} \mathfrak{A}) \preceq (\log k \log^\chi \log k)^3$ has been proved in [9]. The assertion for $d > 1$ follows straightforward by using (26) and (17). □

4.2 The preconditioner for the Schur-complement

We have already mentioned that the Schur complement $S = C + BA^{-1}B^\top$ is also symmetric positive definite. In order to develop an spd-preconditioner for S, let

$$M = \left[\int_\Omega \phi_i^p(x)\phi_j^p(x)\,\mathrm{d}x\right]_{i,j=1}^m \tag{27}$$

be the mass matrix with respect to the basis $[\Phi_p]$. Then, it can be proved that

$$\tilde{\beta}_1^2 M \leq S \leq M$$

for the Stokes equation if $C = 0$, where $\tilde{\beta}_1$ is the inf-sup-constant (9), see e.g. [24]. Since $c(\cdot,\cdot)$ is bounded and positive semidefinite on $\mathbb{Q} \subset L_2(\Omega, \mathbb{R}^d)$, relations (18) and (27) imply

$$\tilde{\beta}_1^2 M \leq S \preceq M. \tag{28}$$

We consider now the case, that the pressure is chosen discontinuous. Then, Legendre polynomials are used as our local basis functions, see (12), (14). Due to the orthogonality relation (13), we introduce the diagonal matrix

$$D = \left[\delta_{ij}\int_\Omega \phi_i^p(x)\phi_i^p(x)\,\mathrm{d}x\right]_{i,j=1}^m \tag{29}$$

as preconditioner for S. Then, the following result can be proved.

Theorem 4.4. *Let D be defined by* (29) *and let S be the Schur complement $S = C + BA^{-1}B^\top$ where A, B, C are defined by* (18)*. Then, the spectral equivalence relation*

$$\tilde{\beta}_1^2 D \preceq S \preceq D$$

holds, where $\tilde{\beta}_1$ is the inf-sup-constant (9).

Proof. The orthogonality of the Legendre polynomials imply $M \sim D$. The assertion follows now from (28). □

4.3 Final condition number estimates

We are now in the position to formulate our final result of this paper. Therefore, we introduce the matrix

$$\hat{T} = \begin{bmatrix} A - \rho\hat{A}_{in,d} & 0 \\ 0 & D \end{bmatrix} \tag{30}$$

where c is some constant, see also(26) and (29) for the definitions of $\hat{A}_{in,d}$ and D, respectively.

Theorem 4.5. *Let T and $\hat{T}$ de defined by* (21) *and* (30)*, respectively. Let K be the discretization matrix* (19) *of the mixed problem* (2) *by using the space pair $\mathbb{V}_k$ and $\mathbb{P}_{k-1}$* (11)*. Then, there exists a $\rho > 0$ such that*

$$\hat{T} \preceq T \preceq (\log k \log^\chi \log k)^3 \hat{T}$$

for any $\chi > 1$. Moreover, the solution of a system with the matrix K (19) *can be performed by using the preconditioned Bramble Pasciak-CG with the preconditioner $\hat{T}$* (30) *in $\mathcal{O}(N(\log N \log^\chi \log N)^{3/2})$ operations.*

Proof. In order to prove the first assertion, we apply theorem 4.1 and verify the assumptions (20). Theorem 4.3 implies that there exists a constant $\rho > 0$ such that $\underline{\gamma}_0 \sim 1$ whereas $(\log k \log^{\chi} \log k)^3 \preceq \overline{\gamma}_0$. Due to lemma 3.1, the inf-sup constant $\tilde{\beta}_1$ is independent of h and k. Finally, theorem 4.4 implies $S \sim D$. Therefore, $\underline{\gamma}_1 \sim 1$ and $\overline{\gamma}_1 \sim 1$. This proves the first assertion. Since a multiplication with each of the involved matrices can be performed in $\mathcal{O}(N)$ operations, see theorem 4.3, the second result follows from the properties of the PCG-method. □

Remark 4.6. *1. In order to apply the Bramble-Pasciak CG, the parameter ρ has to be chosen properly. If ρ is too large the matrix $\hat{T}$ in* (30) *becomes indefinite. If ρ is chosen too small the upper constant $\overline{\gamma}_0$ becomes larger which results in worse condition number estimates in theorem 4.5. Rough estimates of ρ can be obtained by a eigenvalue computations in a precomputing step.*

2. *An alternative to the Bramble-Pasciak-CG is the MINRES algorithm. Then, using the ingredients of lemma 3.1, theorem 4.3 and theorem 4.4, a solution in quasioptimal complexity can also be shown, see [24] for the theoretical background.*

3. *We have derived an quasioptimal solver for the Stokes problem, since the bilinear form $a(\cdot,\cdot)$ in* (3) *is H^1 elliptic. For the linear elasticity problem, the bilinear form $a(\cdot,\cdot)$ in* (5) *is H^1 elliptic. The constants in the second inequality* (6) *do not depend on the choice the Lamé parameters λ and μ. However, the constants depend on geometry of the domain. Since also $c(\cdot,\cdot)$ in* (5) *does not depend on large λ which is the case for nearly incompressible material, we have developed a robust hp-FEM solver for linear elasticity in quasioptimal complexity.*

4. *The results of the theorem remain true if the matrix $\hat{A}_{in,d}$ in* (30) *or equivalently $\hat{A}_{in,1}$ in* (**??**) *is replaced by another quasioptimal solver for the Laplace equation.*

5 Numerical experiments

Finally, some numerical experiments are presented.
In all experiments, the system (19) for the Stokes problem is solved by the Bramble-Pasciak-CG with the preconditioner (30). A relative accuracy of $\epsilon = 10^{-5}$ is chosen. The domain $\Omega = [-1,1]^d$, $d = 2,3$, respectively, is used with pure Dirichlet boundary conditions, e.g. $\partial\Omega = \Gamma_0$. The mesh consists of the union of 2^d, $d = 2,3$ congruent elements.
Since the condition number estimates in Theorem 4.5 depend strongly on the inf-sup constant $\tilde{\beta}_1$ (9), figure 1 shows the behavior of the inf-sup constant on the polynomial degree for different types of single elements. We observe that the value of the inverse of the inf-sup-constant of the Q_k-$P_{k-1,disc}$-element tends to converge to a fixed value. The inverse of the inf-sup-constant of the Q_k-$Q_{k-2,disc}$-element increases as k increases, which coincides with the theory. Nevertheless also in the later case these values are still of moderate size in 2D. However, in 3D these values reach unacceptable values.
Next, the iteration numbers of the preconditioned Bramble-Pasciak-CG are displayed in Figure 2 for the Q_k-$P_{k-1,disc}$-element and Q_k-$Q_{k-2,disc}$-element with discontinuous pressure.
We observe lower iteration numbers for the Q_k-$P_{k-1,disc}$-element, where the inf-sup-constant is independent of the polynomial degrees k, see lemma 3.1. The reason, why the iteration numbers grow moderately in this case, is the wavelet preconditioner for the matrix A, which is optimal only up to some logarithmic term. This is due to Theorem 4.3. We observe higher iteration numbers for the Q_k-$Q_{k-2,disc}$-element. Although the proposed solver is not optimal in complexity for the Q_k-$Q_{k-2,disc}$-element, the iteration numbers of the Bramble-Pasciak-CG differ only slightly in comparison to the iteration numbers of the Bramble-Pasciak-CG for the Q_k-$P_{k-1,disc}$-element if $k \leq 100$. This situation becomes different in 3D where the iterations numbers for the Q_k-$Q_{k-2,disc}$-element are much higher than for the Q_k-$P_{k-1,disc}$-element already for $k \geq 10$.
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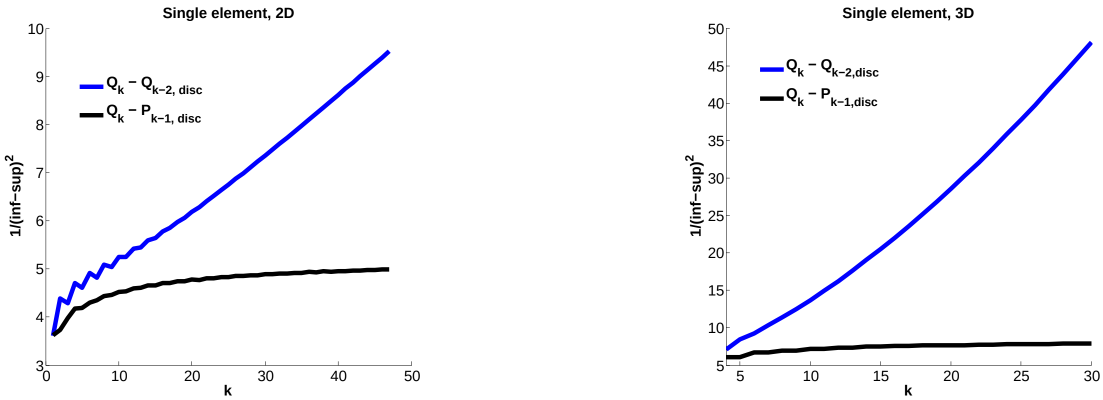

Figure 1: Dependence of the discrete inf-sup constant of several single elements on the polynomial degree k, for $d = 2$ (left) and $d = 3$ (right).

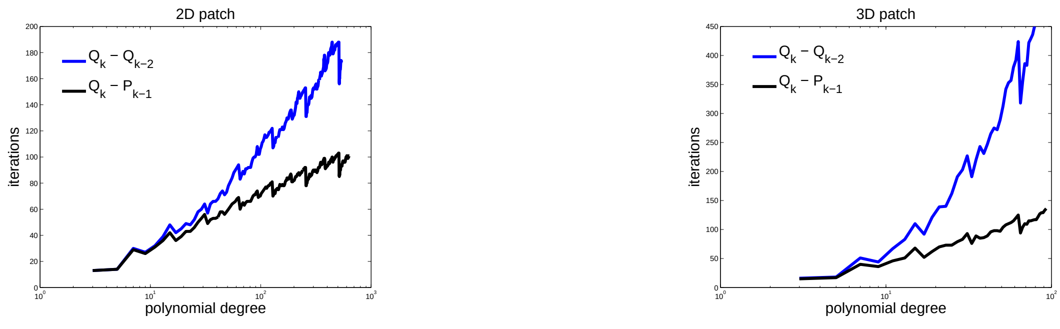

Figure 2: Iteration numbers of the preconditioned Bramble-Pasciak-CG for different elements, $d = 2$ (left), $d = 3$ (right).

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